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MATHEMATICAL MODEL OF CREEP FOR A MICROINHOMOGENEOUS NONLINEARLY ELASTIC MATERIAL

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A mathematical model is proposed to describe the experimentally observed creep effect on the instantaneous elastic deformation of physically nonlinear elastic microinhomogeneous materials. Using a structural model of the medium, it is shown that, during unloading of a sample after creep at constant stress, the elastic strain can be both larger and smaller than the elastic strain during loading. It is shown that calculation results for a biocomposite material are in good agreement with experimental data.

Key words: *nonlinear elasticity, creep, microinhomogeneous medium, structural model, unloading theorem, experimental data.*

Experimental studies [1, 2] of the uniaxial creep of a natural biocomposite material (bone tissue) have shown that the instantaneous elastic strain of this material during loading $e^{\rm load}$ at $t = 0 + 0$ differs significantly from that during complete unloading $e^{\rm unload}$ after creep at constant stress. In this case, both the relation $e^{\rm unload} > e^{\rm load}$ [1] and the relation $e^{\rm unload} < e^{\rm load}$ [2] can hold, i. e., creep strain influences instantaneous elastic strain. An analysis of experimental data [1, 2] shows that bone tissue is a nonlinear elastic material (see also [3, 4]).

At the same time, the possibility of drift of elastic strain due to creep strain has been shown [5] in constructing kinetic creep equations within the framework of control theory; however, conditions under which this effect is possible were not indicated. It has been shown theoretically [6–8] for particular problems for laminar composites [6] and composite microinhomogeneous media [7, 8] that creep influences instantaneous elastic deformation only if even one of the phases of the composite is nonlinear elastic. However, a systematic analysis of this phenomenon has not been performed.

In view of the aforesaid, the purpose of the present work is to develop a mathematical model for the uniaxial creep of a microinhomogeneous nonlinear elastic media that takes into account the effect of creep deformation on the elastic deformation of samples and to validate the model by comparing with experimental data.

1. To elucidate conditions under which instantaneous elastic deformation depends on rheological deformation, we consider the case of uniaxial creep of a natural composite material (bone tissue), for which, as noted above, this relation was confirmed experimentally [1, 2]. Because this material is a two-phase composite material, its deformation properties can be modeled using a simple two-element structural model (a pair of models connected in parallel, such as the Maxwell model), for which the equilibrium and deformation compatibility equations are written as

$$\alpha\sigma_1(t) + (1-\alpha)\sigma_2(t) = \sigma_0(t); \qquad (1)$$

$$\varepsilon_1(t) = \varepsilon_2(t). \qquad (2)$$

Here σ_i and ε_i $(i = 1, 2)$ are the stress and strain in the ith element, respectively, α is the "weight" of the first element of the structural model, and σ_0 is the macrostress applied to the sample. It is assumed that the strains in the local elements ε_i are equal to the strain of the sample ε.

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Because the material is considered nonlinear elastic, each local element of the model has simple deformation properties (nonlinear elasticity and nonlinear viscosity):

$$\varepsilon_i(t) = e_i(t) + p_i(t); \tag{3}$$

$$e_i = \sigma_i|\sigma_i|^{n_i-1}/E_i, \qquad \dot{p}_i = a_i\sigma_i|\sigma_i|^{m_i-1} \quad (i = 1, 2). \tag{4}$$

Theorem 1. *Let the structural model be specified by relations* (1)–(4). *If* $n_1 = 1$ *and* $p_1^* < p_2^*$, *then, the instantaneous elastic strains of the sample during loading* $e^{\rm load}$ *and unloading after creep* $e^{\rm unload}$ *obey the inequality* $e^{\rm load} > e^{\rm unload}$ *for* $n_2 > 1$, *the inequality* $e^{\rm load} < e^{\rm unload}$ *for* $0 < n_2 < 1$, *and the equality* $e^{\rm load} = e^{\rm unload}$ *for* $n_2 = 1$ *[*$p_i^* = p_i(t^*)$, *where* $i = 1, 2$, *is the creep strain accumulated in the local elements of the model by the moment of unloading at* $t = t^*$*)].*

A proof of the theorem for a more general structural model for l ($l > 2$) local elements is given in [8].

It should be noted that by the instantaneous elastic strain e of a sample during loading and unloading at any time $t = t^*$ is meant the quantity

$$e(t^*) = \varepsilon(\sigma_0(t^* + 0)) - \varepsilon(\sigma_0(t^* - 0)). \tag{5}$$

Thus, a change (drift) of the instantaneous elastic strain due to creep deformation of materials is possible only for physically nonlinear elastic materials.

In (4), we set $n_1 = m_1 = 1$ to simplify the further calculations and analysis. Then, from (3) and (4), we have

$$\frac{d\varepsilon_1}{dt} = \frac{1}{E_1}\frac{d\sigma_1}{dt} + a_1\sigma_1, \qquad \frac{d\varepsilon_2}{dt} = \frac{1}{E_2}\frac{d(\sigma_2|\sigma_2|^{n_2-1})}{dt} + a_2\sigma_2|\sigma_2|^{m_2-1}. \tag{6}$$

In the case of instantaneous loading of the sample by stress σ_0 at $t = +0$, Eqs. (1), (2), (5), and (6) imply

$$\alpha\sigma_1^0 + (1-\alpha)\sigma_2^0 = \sigma_0, \qquad \sigma_1^0/E_1 = (\sigma_2^0)^{n_2}/E_2 = e^{\rm load}(\sigma_0),$$

whence it is easy to obtain the relation

$$\alpha E_1 e^{\rm load}(\sigma_0) + (1-\alpha)(E_2 e^{\rm load}(\sigma_0))^{1/n_2} = \sigma_0. \tag{7}$$

Here $\sigma_i^0 = \sigma_i(+0)$ and $e^{\rm load}(\sigma_0)$ is the nonlinear dependence of the instantaneous elastic strain of the sample on the applied stress σ_0 at $t = 0$.

In Eq. (7), if E_1, E_2, and α are assumed to be known, this is an equation for determining the coefficient n_2.

To determine the rheological coefficients a_1, a_2, and m_2, it is necessary to consider the limiting state of the structural model that corresponds to steady-state creep. In addition, it is necessary to know the dependence of the steady-state creep rate of the material sample $\dot{\varepsilon}'$ on stress. This dependence is specified as $\dot{\varepsilon}'(\sigma_0) = a\sigma_0^m$. Then, from Eqs. (1), (2), (6), and (7), we obtain

$$\alpha\sigma_1' + (1-\alpha)\sigma_2' = \sigma_0, \qquad a_1\sigma_1' = a_2(\sigma_2')^{m_2} = a\sigma_0^m, \tag{8}$$

where σ_1' and σ_2' are the stresses in the elements of the structural model that correspond to the steady-state creep stage.

From Eqs. (8), it follows that

$$\alpha\,\frac{a}{a_1}\,\sigma_0^m + (1-\alpha)\Big(\frac{a}{a_2}\,\sigma_0^m\Big)^{1/m_2} = \sigma_0. \tag{9}$$

If α, a_1, and a_2 are assumed to be known, Eq. (9) can be considered as an equation for m_2.

We consider the unloading of the sample at $t = t^*$ ($\sigma_0 = 0$ at $t > t^*$). From Eqs. (1), (2), and (6), we have

$$\alpha\sigma_1^* + (1-\alpha)\sigma_2^* = 0, \qquad \sigma_1^*/E_1 + p_1^* = \sigma_2^*|\sigma_2^*|^{n_2-1}/E_2 + p_2^*,$$

whence we obtain the following equations for the stresses σ_1^* and σ_2^*:

$$\frac{\sigma_1^*}{E_1} + \frac{\alpha}{1-\alpha}\,\frac{\sigma_1^*|\sigma_1^*|^{n_2-1}}{E_2} + p_1^* - p_2^* = 0, \qquad \sigma_2^* = -\frac{\alpha}{1-\alpha}\,\sigma_1^*.$$

Here $\sigma_i^* = \sigma_i^*(t^* + 0)$, p_i^* ($i = 1, 2$) is the creep strain accumulated in the ith local element of the structural model by the moment of unloading.

TABLE 1

Experimental and Calculated Elastic Strains
of Biocomposite Material during Loading and Unloading after Creep

σ_0, MPa	e^{load}, %		e^{unload}, %	
	Experiment	Calculation for structural model	Experiment	Calculation for structural model
35.71	0.178	0.178	0.152	0.162
53.56	0.255	0.250	0.245	0.236
74.56	0.470	0.445	0.670	0.642
90.41	0.570	0.570	—	—

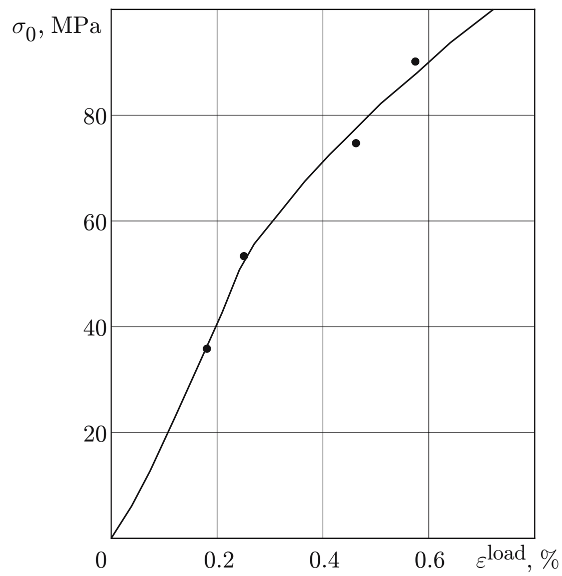

Fig. 1. Diagram of instantaneous elastic strain of bone tissue: the solid curve refers to calculation results, and the points refer to the experimental data.

Once σ_1^* and σ_2^* are determined, the instantaneous elastic strain of the sample can be found from the formula

$$e^{\text{unload}} = (\sigma_1(t^* - 0) - \sigma_1^*)/E_1.$$

The proposed structural model allows a numerical calculation of creep using Eqs. (1), (2), and (4). For this, the time of observation of the creep strain is divided by points $t = 0, t_1, t_2, \ldots$ into intervals $t \in [t_j, t_{j+1}]$, in each of which the system of differential equations is solved numerically (for example, using the Euler method).

2. To check the validity of the results obtained on the basis of the structural model (1)–(4), we use the experimental data of [1, 2].

In experiments [2], a biocomposite material (bone tissue) was tested in the region of linear creep (more precisely, weak nonlinearity) at $\sigma_0 = 35.71$ and 53.56 MPa and a loading duration $\tau = 100$ min with subsequent unloading. In [1], the region of nonlinear creep of this material was studied at $\sigma_0 = 74.56$ MPa and $\tau = 200$ min with subsequent unloading and at $\sigma_0 = 90.41$ MPa up to fracture.

Experimental instantaneous elastic strains during loading and unloading [1, 2] are given in Table 1. If one constructs a curve of the instantaneous strain of bone tissue $\sigma_0 \sim e^{\text{load}}$ (the points in Fig. 1), it is evident that it has a nonlinear nature; in the linear creep region ($0 \leqslant \sigma_0 \leqslant 53.56$ MPa), the curve should be convex downward, and in the nonlinear creep region (53.56 MPa $\leqslant \sigma_0 \leqslant 90.41$ MPa), it should be convex upward. This shape of the of instantaneous elastic strain curve of bone tissue is supported by the experimental data of [3].

In view of the aforesaid, we used a piecewise nonlinear approximation of the experimental data on elastic deformation at the macrolevel:

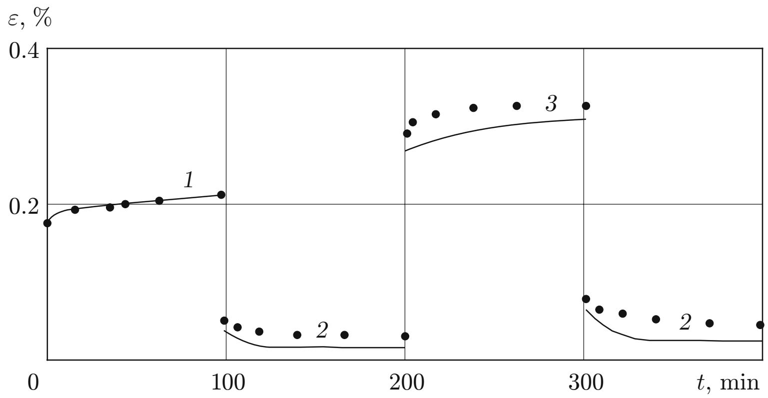

Fig. 2

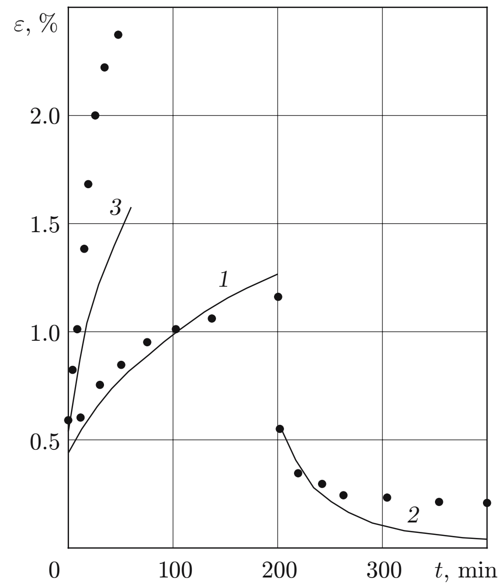

Fig. 3

Fig. 2. Strain of bone tissue versus time in the linear creep region: the points refer to experimental data, and the solid curves refer to calculations using the structural model at $\sigma_0 = 35.71$ (1), 0 (2), and 53.56 MPa (3).

Fig. 3. Strain of bone tissue versus time in the nonlinear creep region: the points refer to experimental data, and the solid curves refer to calculations using the structural model at $\sigma_0 = 74.56$ (1), 0 (2), and 90.41 MPa (3).

$$e^{\text{load}}(\sigma_0) = \sigma_0|\sigma_0|^{n-1}/E \tag{10}$$

(the solid curve in Fig. 1). In this case, $E = 1.302 \cdot 10^4$ (MPa)^n and $n = 0.88$ for $0 \leqslant \sigma_0 \leqslant 53.56$ MPa, and $E = 32.106 \cdot 10^4$ (MPa)^n and $n = 1.685$ for 53.56 MPa $\leqslant \sigma_0 \leqslant 90.41$ MPa. The approximation was constructed using the least-squares method, so that the plot passed through the point at $\sigma_0 = 53.56$ MPa.

According to Eq. (10), since $n < 1$, the initial elastic modulus (at $\sigma_0 = 0$) vanishes:

$$\frac{d\sigma_0}{de^{\text{load}}} = \frac{1}{(n/E)\sigma_0^{n-1}}, \qquad \frac{d\sigma_0}{de^{\text{load}}}\Big|_{\sigma_0=0} = 0.$$

However, one should bear in mind that, for a nonlinear approximation of the form (10), the coefficient E is not the elastic modulus in the classic sense but it is one of the two parameters used to describe the nonlinear nature of the instantaneous elastic strain curve. In relation (10), the quantity E is in (MPa)^n.

Figures 2 and 3 give experimental dependences of the total strain on time (points) [1, 2]. To determine the parameters a and m in the second relation (8), one should know the steady-state creep rate of the bone tissue sample, which was determined in both the linear (Fig. 2) and nonlinear (Fig. 3) creep regions by the last three experimental points in each loading stage. The values of the rheological characteristics a and m were found using the least-squares method. For $0 \leqslant \sigma_0 \leqslant 53.56$ MPa, we obtained $a = 1.526 \cdot 10^{-6}$ (MPa)$^{-m} \cdot \text{min}^{-1}$ and $m = 1$, and for 53.56 MPa $\leqslant \sigma_0 \leqslant 90.41$ MPa, $a = 2.05 \cdot 10^{-19}$ (MPa)$^{-m} \cdot \text{min}^{-1}$ and $m = 8.455$.

The constants α, E_1, E_2, a_1, and a_2 of the structural model (1)–(4) were determined from the minimum condition for the objective function

$$Z = \sum_{k=1}^{s} \Big(\frac{\varepsilon(t_k) - \varepsilon^*(t_k)}{\varepsilon^*(t_k)}\Big)^2. \tag{11}$$

Here $\varepsilon(t_k)$ is the total strain during loading at the time $t = t_k$ calculated for the structural model, and $\varepsilon^*(t_k)$ is the experimental total strain at the same time. It should be noted that formula (11) contains the instantaneous elastic strains during both loading and unloading.

In the linear creep region ($0 \leqslant \sigma_0 \leqslant 53.56$ MPa), we varied the coefficients α, E_1, E_1/E_2, and a_1. The value of n_2 was determined from Eq. (7). Since, in the linear creep region, $m_2 = 1$, the coefficient a_2 was found from Eqs. (9). The following values of the parameters of structural model were obtained: $\alpha = 2/9$, $E_1 = 9350$ MPa, $E_2 = 12{,}290\ (\mathrm{MPa})^{n_2}$, $n_2 = 0.83$, $a_1 = 4.96 \cdot 10^{-6}\ \mathrm{min}^{-1} \cdot (\mathrm{MPa})^{-1}$, and $a_2 = 1.24 \cdot 10^{-8}\ \mathrm{min}^{-1} \cdot (\mathrm{MPa})^{-1}$.

In the nonlinear creep region (53.56 MPa $\leqslant \sigma_0 \leqslant$ 90.41 MPa), we varied the values of E_1, E_1/E_2, a_1, and a_2 (the value α was determined beforehand). The value of n_2 was determined from Eq. (7), and m_2 from Eq. (9). The following values of the parameters of the structural model in this region were calculated: $E_1 = 3.72 \cdot 10^4$ MPa, $E_2 = 2.628 \cdot 10^5\ (\mathrm{MPa})^{n_2}$, $n_2 = 1.82$, $m_2 = 8.82$, $a_1 = 7.87 \cdot 10^{-7}\ \mathrm{min}^{-1} \cdot (\mathrm{MPa})^{-1}$, and $a_2 = 7.475 \cdot 10^{-23}\ \mathrm{min}^{-1} \cdot (\mathrm{MPa})^{-m_2}$.

In Figs. 2 and 3, the solid curves show the results of calculations using the structural model (1)–(4) for the total strain of the bone tissue. Calculated instantaneous elastic strains during loading e^{load} and unloading e^{unload} are given in Table 1.

From the data given in Figs. 2 and 3, it follows that the proposed structural model adequately describes both the creep strain of the bone tissue and the variation of the instantaneous elastic strain due to creep strain (see Table 1). The significant difference between the calculated and experimental data in Fig. 3 at $\sigma_0 = 90.41$ MPa is apparently due to the fact that, in this case, deformation occurs in the prefracture region and this regime differs significantly from the other loading regimes considered.

3. An analysis of the results obtained in the numerical experiments leads to the following conclusions.

For nonlinearly elastic materials described by the model (1)–(4), there is a drift of the diagram of the elastic strain $e \sim \sigma_0$ due to creep and redistribution of microstresses in the elements of the structural model; all diagrams are located between the two diagrams corresponding to the stationary states: the diagram of $e \sim \sigma_0$ at $t = 0$ and the diagram corresponding to the behavior of the structural model under load as $t \to \infty$.

Instantaneous elastic deformation has the mechanical memory properties, because the initial elastic properties of samples are completely recovered in the creep process during unloading ($t \to \infty$), and, simultaneously, it has viscoelasticity properties, because the instantaneous elastic strain explicitly depends on time.

For the creep deformation of a sample with exposures at constant stresses, the elastic strain diagram has specific hysteresis loops. The factors responsible for the complex behavior during elastic deformation are stress redistribution in the local elements of the structural model during loading and stress relaxation during unloading, the nonuniform accumulation of creep strain in the elements and, finally, nonlinearity in the elastic deformation of these elements.

The elastic creep strain diagram is similar to elastic strain diagrams at constant stress during aging [9], and the hysteresis loops appearing in the diagram during loading and unloading after creep are similar to the hysteresis loops in strain diagrams of materials exhibiting the mechanical shape memory effect (see, for example, [10]). However, the mechanism of the phenomenon considered is different and occurs only for nonlinear elastic materials.

Thus, the change (drift) of instantaneous elastic deformation due to creep deformation was described within the framework of the proposed model using bone tissue as an example. The conditions leading to this change were found, and this was shown to be possible only for nonlinear elastic materials.

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